Minimal Immersions, Einstein's Equations and Mach's Principle

BIRGER NIELSEN

Mathematical Institute
The Technical University of Denmark
Building 303 - DK-2800 Lyngby
Denmark

Abstract. *A geometrical stress energy tensor for semi-Riemannian manifolds is described and a Mach's principle is formulated. It is shown that vacuum occurs if and only if the manifold is a totally geodesic submanifold of a flat semi-Euclidean space. Furthermore the Einstein equations are attained with the cosmological constant appearing as the mean curvature of an isometric immersion. A minimal submanifold of a semi-Euclidean space can thereby be regarded as a solution to Einsteins equations without a cosmological constant. Intrinsic conditions that will allow a 4-dimensional semi-Riemannian manifold to be immersed isometrically into 5-dimensional semi-Euclidean space as a minimal hypersurface are found. From this result it is possible to find explicit minimal hypersurfaces of Robertson-Walker type in a 5-dimensional Minkowski space and it is observed that they all contain an initial singularity.*

1. INTRODUCTION

In Einstein initial formulation of the general theory of relativity, he stated three fundamental principles on which any theory of gravity should rest.

1) The principle of general covariance.
2) The principle of equivalence.
3) Mach's principle.

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The two first principles have clear physical ramifications and are well understood in the language of differential geometry. Nowadays they are usually interpreted in the following way:

The theory should be formulated invariantly on a spacetime, which is a four-dimensional differentiable manifold (principle 1.). The curvature of this spacetime, derived from a metric tensor field, is interpreted as the action of gravity (principle 2.). The metric tensor field can then be thought of as a potential for the gravitational interaction.

The third principle, Mach's principle, has however an apparently very vague status, and is still a subject to discussion among scientists in relativity. The original idea of Mach was to abolish Newton's concept of an absolute space, as required in the law of inertia. An inertial system should no longer be defined relative to an absolute space but relative to the fixed stars of the external galaxies. This is not only more satisfying from a philosophical point of view, but also in agreement with observations in astronomy. Mach took this agreement as an indication of an intimite connection between the very distant matter in the universe and the concept of inertia. Remove the fixed stars and the concept of inertial mass has no meaning. Inertial mass should therefore not be thought of as an intrinsic property, but as depending on a background matter.

Mach never proposed any explicit quantitative scheme for his new interpretation of the law of inertia. He in fact never formulated it as a principle. This was first done by Einstein in 1918, [5] in an attempt to formulate a consistent theory of relativity with the above mentioned principles on equal footing. Einstein gives the idea of Mach a concrete form by requiring that the matter distribution of the universe should completely determine the geometry of spacetime (see also [6]). The distribution of energy and matter is related to the geometry of spacetime via the so-called Einstein equations, and it was the initial hope of Einstein that this determinacy in fact already was incorporated in these equations. This turned out to be false. Essentially different metric solutions to the Einstein equations were found, that describe the same matter distribution. Einstein's initial enthusiasms for Mach's principle therefore waned in his later years and finally vanished. In 1954 he writes to a colleague, «as a matter of fact, one shoud no longer speak on Mach's principle at all». [7].

The question which is raised by the Mach's principle concerns the origin of the initial mass and has as such deep connection with one of todays basic unsolved problems in theoretical physics – the quark masses. There exists no theory, at present, which can give any indications of the quark masses. A final understanding and formulation of Mach's principle should therefore also involve quantum field theory.

In this paper I shall give a new geometrical formulation of the Mach's principle, that leads to consider classical interesting geometrical questions rather than solving fundamental physical problems. The main objects are immersions of semi-Riemannian manifolds. In particular the case of four dimensional Lorentzian manifold immersed in a flat five dimensional pseudo-Euclidean space. A cosmological solution to Einstein's equations of Robertson-Walker type is derived as a minimal hypersurface.

2. EINSTEIN'S EQUATIONS AND MINIMAL IMMERSIONS

In the spirit of Einstein I shall in the following define a geometrical Mach's principle for semi-Riemannian manifolds.

Let (M, g) denote a semi-Riemannian manifold of dimension m with metric tensor field g. We consider the geometry of (M, g) in the frame of classical field theory.

The differentiable manifold M has a natural geometrical field, given by a map, γ, that to each point $p \in M$ assigns the tangent space T_pM at p, i.e.

$$\gamma : p \mapsto T_pM.$$

Connected with this map we define a Lagrange function $L(\gamma)$ on the field and an action integral:

$$I = \int_M L(\gamma) \, \mathrm{d\,vol}.$$

The equation of motion or the field equations are derived from an action principle, i.e. we assume that the field chosen by nature is a field extremizing this action integral. Thus stationary action under variation of the map γ leads to the Euler-Lagrange equations i.e. equations of motions for the field.

To obtain a set of field equations for the metric tensor field g by this action principle, we need to couple the field to the metric tensor field g by defining the Lagrange function in the following way.

Consider (M, g) isometrically immersed into a flat semi-Euclidean space E^n by an immersion:

$$\phi : M \to E^n,$$

where the tangent space T_pM is identified with the subspace $T_{\phi(p)}\phi(M)$ of E^n. Also since the discussions here are local, $\phi(M)$ will be identified with M.

The field γ now appears naturally by assigning to each point p the tangent space T_pM at $\phi(p)$ translated to the origin of E^n. Hence we have a map:

$$\gamma : M \to G_{m,n}$$

into the Grassmannian $G_{m,n}$ of m-dimensional oriented subspaces of E^n. Classically this map is called the Gauss map, associated to the immersion ϕ.

On $G_{m,n}$ we have a canonical metric k induced from the metric on E^n. A simple geometrical Lagrange function involving this metric and the metric g is the following:

$$e(\gamma) = \frac{1}{2}\, \partial_i \gamma^\alpha \partial_j \gamma^\beta g^{ij} k_{\alpha\beta} = \frac{1}{2} \langle d\gamma, d\gamma \rangle = \frac{1}{2} \text{ trace } \gamma^* k, \tag{1}$$

where $\langle . \,, . \rangle$ denotes the induced metric from g on the tensor bundle over M. The action is then given by:

$$I = \int_M e(\gamma) \text{ d vol.} \tag{2}$$

This action also appears in a number of different fields theory, e.g. in a generalization of Heisenbergs classical theory of ferromagnetism, in the nonlinear σ-model in particle physics (for references see [11]), and recently also in Polyakov's string theory [15]. In mathematics the action appears in the theory of harmonic maps and its stationary values are called harmonic maps [4]. The corresponding Euler-Lagrange equations are

$$-\Delta \gamma^\alpha + \Gamma^\alpha_{\beta\gamma} \partial_i \gamma^\beta \partial_j \gamma^\gamma g^{ij} = 0, \tag{3}$$

where Δ is the Laplace-Beltrami operator on M and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connection on $G_{m,n}$. In order to obtain the field equations in the metric field g, however, we consider variations in g in this action. Hereby we obtain the so-called canonical stress energy tensor T, defined by the classical variational expression:

$$\delta I = \int_M T^{ij} \delta g_{ij} \text{ d vol.}$$

In connection with harmonic maps the 2-tensor $T = T^{ij}$ has been studied in [1]. They found the following expression for the stress energy tensor:

$$T_\gamma = e(\gamma) g - \gamma^* k. \tag{4}$$

In a variational formulation one adds to the initial action a matter contribution, and then demand the total action stationary under variation in the metric g. Hence we consider equation (4) as a definition of the stress energy momentum caused by geometry.

Since it is always possible isometrically to immerse a semi-Riemannian manifold into semi-Euclidean space by a theorem of Clarke [3], (a generalization of the result [10]), we are always given a stress energy tensor on a semi-Riemannian manifold by the above procedure. The tensor is not unique, but depends on the chosen immersion. However, if we assume the dimension of M to be 4 and the codimension of the immersion less or equal to 3 it will be unique up to a rigid motion by a theorem of Berger, Bryant and Griffiths [2].

We can now formulate a geometrical Mach's principle by the following question.

Mach's principle: Let g and g' be two semi-Riemannian metrics on M and γ, γ' be their corresponding Gauss maps. If $T_\gamma = T_{\gamma'}$, are they then isometric?

In the special case of $T_\gamma = T_{\gamma'} = 0$ we have an affirmative answer by the following theorem.

THEOREM 1. *Let (M, g) be a semi-Riemannian manifold of dimension $m > 2$ and let γ be the Gauss map associated to a non-degenerate isometric immersion $\phi : M \to E^n$.*

Then $T_\gamma = 0$ if and only if (M, g) is a totally geodesic submanifold of E^n.

Proof. Assume $T_\gamma = 0$ then by the definition:

$$e(\gamma)g = \gamma^* k$$

and taking trace of both sides of this equation yields $me(\gamma) = e(\gamma)$ and by hypothesis we have

$$e(\gamma) = \frac{1}{2} \langle d\gamma, d\gamma \rangle = 0.$$

In the Riemannian case we have $d\gamma = 0$, i.e. γ is a constant map. In the semi-Riemannian case the assumption of a non-degenerate immersion is needed. Consider the second fundamental form defined by:

$$\beta_\phi = \nabla^E - \nabla$$

where ∇^E is the flat connection on E^n and ∇ the Levi-Civita connection on M respectively. The second fundamental form is a symmetric 2-tensor with values in the normal bundle $TM^\perp$. We say that the immersion is non-degenerate if and only if:

$$\langle \beta_\phi, \beta_\phi \rangle = 0 \Longleftrightarrow \beta_\phi = 0.$$

The differential $d\gamma : TM \to TG_{m,n}$ is a map with values in the tangent bundle of the Grassmannian. Identifying this bundle with the tensor decomposition: $K^* \otimes K^{\perp}$, we have the basic relation $d\gamma = \beta_\phi$. Here K denotes the tautological bundle over the Grassmannian, i.e. the bundle where we over each m-plane assign the plane itself. Hence we see $d\gamma = \beta_\phi = 0$, so (M, g) is a totally geodesic submanifold of E^n.

If we on the other hand assume $\beta_\phi = 0$ this implies $d\gamma = 0$ hence by (2) $T_\gamma = 0$. ∎

An explicit expression for the stress energy tensor is given by the following result.

THEOREM 2. *Let $\gamma : M \to G_{m,n}$ be the Gauss map associated to a minimal isometric immersion $\phi : M \to E^n$. Then the field equations for the metric field are given by:*

(5) $$T_\gamma = Ric - \frac{1}{2} Sg,$$

where Ric is the Ricci curvature and S is the scalar curvature on (M, g).

Proof: This is an application of the Gauss equations. Let R denote the Riemannian curvature tensor, then:

(6) $$\langle R(X, Y)Z, W\rangle = \langle \beta(X, W), \beta(Y, Z)\rangle - \langle \beta(X, Z), \beta(Y, W)\rangle,$$

relating the curvature of the manifold (M, g) to the second fundamental form of the immersion. Here X, Y, Z, W are vectorfields on M. The mean curvature vector h is defined by

(7) $$h = \operatorname{trace} \beta.$$

The immersion is called minimal if the mean curvature vector vanish everywhere. A direct computation using the Gauss equations yields a result of Obata [13].

(8) $$\gamma^* k = \langle \beta, h\rangle - Ric.$$

By taking trace we arrive at the following expression for the Lagrange function:

$$e(\gamma) = -\frac{1}{2}(S - \langle h, h\rangle).$$

This is in fact identical with the usual Lagrange functional in general relativity (see [8]) if we interpret $\langle h, h\rangle$ as a cosmological constant. Using formula (3) with $h = 0$ we find (5). ∎

REMARKS. Under the hypothesis that the immersion is minimal we have the following consequences:

i) The stress energy tensor has no explicit dependency on the immersion. It is given uniquely by the intrinsic geometry of (M, g).

ii) The field equations reduces to the classical Einstein equations without cosmological constant (equation 5).

iii) The Gauss Map γ is an extremum of the action integral (2) by a theorem of Ruh and Vilms [16].

iv) The usual approach to the problem of general relativity can be viewed as a generalization of the classical Plateau problem for minimal surfaces in the following way:

Given a symmetric 2-tensor T on a semi-Riemannian manifold (M, g), does there exist an isometric minimal immersion $\phi : M \to E^n$ with T as its stress energy tensor?

3. MINIMAL HYPERSURFACES

The general problem stated in remark iv) is unfortunately very difficult to tackle; but since a number of exact solutions to the Einstein equations are already known, one could instead consider the following more feasible problem: Can any of the known exact solutions to Einsteins equations be realized as a minimal submanifold of a flat semi-Euclidean space?

Due to the result of Theorem 1. however, one can immediately exclude all vacuum solutions, such as Schwartzchild, Kerr- and plane waves solutions, since $T = 0$ implies flat geometry. One could therefore say that the level of validity of the Einstein equations, in this picture, is on the macroscopic level, i.e. only cosmological solutions has to be considered.

To simplify matters further we will restrict ourselves to the case of hypersurfaces, i.e. immersions into E^5, and pose the question: given a 4-dimensional Lorentzian manifold (M, g) what is the necessary and sufficient condition for (M, g) to be immersed as a minimal hypersurface of E^5? We have the following local answer to this problem:

THEOREM 3. *Let (M, g) be a 4-dimensional semi-Riemannian manifold and assume that the Ricci curvature Ric, viewed as self adjoint transformation, has only real non-positive eigenvalues and is of constant rank r in an open set $U \subset M$.*

Then a neighbourhood U' of a point p in U can be minimally and isometrically *immersed into* E^5 *if there exists an orthonormal frame* (e_i) *diagonalizing the Ricci transformation in* U *and such that the sectional curvatures* $K(e_i \wedge e_j)$ *in that frame satisfy one of following conditions:*

i) If $r = 4$: (setting $K_{ij} = K(e_i \wedge e_j)$)

$$K_{12} = K_{13}, \; K_{24} = K_{34}, \; K_{23} = -\frac{1}{2}(K_{12} + K_{34}),$$

$$K_{14} = -2\,\frac{K_{12}K_{34}}{K_{12} + K_{34}}.$$

ii) If $r = 3$:

$$K_{12} = K_{13} = K_{14} = 0 \text{ and } K_{23} = -\frac{K_{24}K_{34}}{K_{24} + K_{34}}$$

iii) If $r = 2$:

$K_{ij} = 0$, except one, e.g. $K_{34} \neq 0$

iv) If $r = 1$ or $r = 0$

all $K_{ij} = 0$.

Furthermore, if $U = M$ *and* M *is a connected and simply connected manifold, then the immersion can be extended to a global immersion.*

Proof. We have to prove that there exists a tensor field A of type $(1,1)$ on M satisfying the equations of Gauss and Codazzi. Then by the fundamental theorem for hypersurfaces, (see [9] and the proof therein), there exists an isometric immersion of a neighbourhood U' of p with A as the second fundamental form. Then we only need to show that the conditions also imply that Trace $A = 0$, i.e. the immersion is minimal.

We shall use the formalism of bivectors. Let $\Lambda^2 T_pM$ be the vectorspace of bivectors spanned by $\{e_i \wedge e_j\}$, an orthonormal basis in the induced inner product on $\Lambda^2 T_pM$ defined by

(10) $$\langle u_1 \wedge v_1, u_2 \wedge v_2 \rangle = \det[\langle u_i, v_j \rangle], \text{ for } u_i, v_j \in T_pM.$$

A linear map $A : T_pM \to T_pM$ defines a linear map $A \wedge A$ on $\Lambda^2 T_pM$ by the relation $A \wedge A(x \wedge y) = Ax \wedge Ay$.

The curvature tensor can then be thought of as a self adjoint linear map $R : \Lambda^2 T_pM \to \Lambda^2 T_pM$ via the relation

$$\langle R(x \wedge y), u \wedge v \rangle = \langle R(x, y)\, v, u \rangle$$

where the R on the right hand side denotes the usual curvature operator on TM.

Let A denote the second fundamentalform, viewed as a self adjoint linear transformation on the tangent space T_pM, then the Gauss and Codazzi's equations take the form:

(11) $$R = A \wedge A$$

(12) ∇A is symmetric.

Let us first investigate the Gauss equations, (11).

The Ricci transformation is defined by the trace:

(13) $$\langle Ric\, X, Y\rangle = \text{trace}\,\{Z \to R(Z, X)\; Y\}$$

or in a orthonormal frame:

$$\langle Ric\, e_i, e_j\rangle = \sum_k \epsilon_k\, \langle R(e_k, e_i)e_j, e_k\rangle$$

$$= \sum_k \epsilon_k\, \langle R(e_k \wedge e_i), e_k \wedge e_j\rangle,$$

where ϵ is the signature of the metric g.

Now, if we apply the Gauss equations we get:

$$\langle Ric\, e_i, e_j\rangle = \sum_k \epsilon_k\, \langle A \wedge A(e_k \wedge e_j), e_k \wedge e_j\rangle$$

$$= \sum_k \epsilon_k\, \langle A\, e_k \wedge A\, e_i, e_k \wedge e_j\rangle$$

$$= \sum_k \epsilon_k\, \langle A\, e_k, e_k\rangle \langle A\, e_i, e_j\rangle - \sum_k \epsilon_k\, \langle A\, e_k, e_j\rangle\langle A\, e_i, e_k\rangle$$

$$= \langle A\, e_i, e_j\rangle\, \text{trace}\, A - \langle A^2 e_i, e_j\rangle.$$

So

(14) $$Ric\, X = AX\, \text{trace}\, A - A^2 X.$$

Since the immersion is minimal if and only if trace $A = 0$, the second fundamental form has to satisfy:

(15) $$A^2 = -\; Ric.$$

The orthonormal frame (e_i) diagonalizes the Ricci transformation. In that frame we construct A as a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, satisfying:

$$\lambda_i^2 = -R_i^i.$$

Now the sectional curvatures are defined by:

$$K(x \wedge y) = \frac{\langle R(x \wedge y), x \wedge y\rangle}{\langle x \wedge y, x \wedge y\rangle}$$

and setting $K(e_i \wedge e_j) = K_{ij}$ the Gauss equations take the form $K_{ij} = \lambda_i\lambda_j$. The Ricci tensor is also given in terms of the sectional curvatures:

$$R_{ii} = \langle Ric\, e_i, e_i\rangle = \epsilon_i \sum_{i \neq j} K_{ij}.$$

Furthermore, the invariant det $A = \lambda_1\lambda_2\lambda_3\lambda_4$ is also by the Gauss equation equal to $K_{12}K_{34} = K_{13}K_{24} = K_{23}K_{14}$. Hence, the eigenvalues of the second fundamental form has to satisfy the following 3 conditions:

(17) $$K_{ij} = \lambda_i\lambda_j$$

(18) $$\begin{aligned} -\lambda_1^2 &= K_{12} + K_{13} + K_{14} \\ -\lambda_2^2 &= K_{12} + K_{23} + K_{24} \\ -\lambda_3^2 &= K_{13} + K_{23} + K_{34} \\ -\lambda_4^2 &= K_{14} + K_{24} + K_{34} \end{aligned}$$

(19) $$K_{12}K_{34} = K_{13}K_{24} = K_{14}K_{23}$$

By comparing $K_{12}^2 = \lambda_1^2\lambda_2^2$ and $K_{13}^2 = \lambda_1^2\lambda_3^2$ one finds either $K_{24} = K_{34}$ or $K_{12} + K_{13} + K_{14} = -\lambda_1^2 = 0$.

Let us first assume Ric is of rank 4, then $K_{24} = K_{34} \neq 0$.

From condition (19) it then follows $K_{12} = K_{13}$, hence by (17) $\lambda_2 = \lambda_3$.

Then

$$K_{23} = \lambda_2\lambda_3 = \lambda_2^2 = -K_{12} - K_{23} - K_{24} \Rightarrow$$

$$K_{23} = -\frac{1}{2}(K_{12} + K_{34})$$

and by (19)

$$K_{14} = -2\,\frac{K_{12}K_{34}}{K_{12}+K_{34}}$$

Now consider the case *Ric* is of rank 3, i.e. one of the eigenvalues for A is zero. We can always assume that $\lambda_1 = 0$. Then $K_{12} = K_{13} = K_{14} = 0$ by the Gauss equation and (18) reduces to

$$\begin{aligned} -\lambda_2^2 &= K_{23} + K_{24} \\ -\lambda_3^2 &= K_{23} + K_{34} \qquad (20) \\ -\lambda_1^2 &= K_{24} + K_{34} \ . \end{aligned}$$

These equations satisfy (17) if

$$K_{23} = \frac{K_{24}K_{34}}{K_{24}+K_{34}} \ .$$

Now assume that also $\lambda_2 = 0$ then $K_{23} = K_{24} = 0$ and (20) reduces to $-\lambda_3^2 = -\lambda_4^2 = K_{34}$. Clearly if also $\lambda_3 = 0$ or $Ric = 0$ then all $K_{ij} = 0$.

Now we consider the Codazzi equations:

$$(\nabla_X A)Y = (\nabla_Y A)X \qquad (21)$$

since $Ric = -A^2$ this clearly implies

$$(\nabla_X Ric)Y = -(\nabla_{AY} A)X. \qquad (22)$$

Now the converse also holds. Assume

$$\nabla_X A^2 Y = \nabla_{AY} AX.$$

If A is of rank 4 then the Codazzi equations follow by setting $Z = AY$.

Then consider the degenerate case, when A has two zero eigenvalues and assume e_1, e_2 are the corresponding eigenvectors. Then any vector Z in T_pM can be written: $Z = AY + z_1e_1 + z_2e_2$ for some $Y \in T_pM$ and $(z_1, z_2) \in R^2$.

Then we have:

$$\begin{aligned} \nabla_X AZ &= \nabla_X A(AY + z_1e_1 + z_2e_2) = \nabla_X A^2 Y = \nabla_{AY} AX \\ &= \nabla_Z AX - z_1\nabla_{e_1} AX - z_2\nabla_{e_2} AX. \end{aligned}$$

So clearly the Codazzi equations hold on the subspace $(z_1, z_2) = (0, 0)$. We need to show that $\nabla_{e_k} AX = 0$ for $k = 1,2$, or in coordinates $\nabla_{e_k} Ae_i = \lambda_{i;k} = 0$. Since $R^i_i = -\lambda_i^2 \Rightarrow R^i_{i;1} = 2\lambda_i\lambda_{i;1}$. Clearly $\lambda_{i;1} = 0$ for $i = 1,2$ since $\lambda_i = 0$ in U for $i = 1,2$. Hence $R^i_{i;1} = 0$ implies $A^i_{i;1} = 0$.

The contracted Bianchi identities are:

$$R^i_{\ell;k} - R^i_{k;\ell} = \sum R^{ir}_{k\ell;r} .$$

Then by the Gauss equations we have:

$$-R^i_{i;1} = \sum_r R^{ir}_{i1\,;r} = \sum_r A^i_i A^r_{1\,;r} = A^i_i A^1_{1;1} + A^i_i A^2_{1;2} + \sum_r A^i_i A^r_{r;1} = 0.$$

The first term is zero since $A^1_1 = \lambda_1 = 0$. The second term is zero since A has ranke 2 in U. The third term is zero since Trace $A = 0$. Similarly prove that $R^i_{i;2} = 0$.

Now assume $\nabla_X Ric\ Y = -\nabla_{AY} AX$ or in coordinates:

$$R^i_{k;\ell} = \sum - A^r_k A^i_{\ell;r} = -\sum (A^r_k A^i_\ell)_{;r} \quad \text{since} \quad \sum_r A^r_{r;k} = 0$$

$$= \sum_r (A^i_k A^r_\ell - A^r_k A^i_\ell)_{;r} - \sum_r (A^i_k A^r_\ell)_{;r}.$$

Then using the Gauss equations:

$$= \sum R^{ir}_{k\ell;r} + R^i_{\ell;k}.$$

But this is just the contracted Bianchi identities.

The finishes the proof of Theorem 3. ∎

Remark. As seen in the proof, the result does not depend on the signature of the metric. Thus the theorem still holds in the case of a Riemannian manifold, now with the redundant assumption of real eigenvalues of the Ricci transformation.

4. APPLICATION TO COSMOLOGY

We will now consider one of the most renowned cosmological solutions to Einstein's equation, namely the Robertson-Walker spacetime. By applying Theorem 3 we shall show how this spacetime can be realized as a minimal submanifold on E^5.

A Robertson-Walker spacetime is most conveniently defined in terms of warped products (see [14]) in the following way. Let S be a connected 3-dimensional Riemannian manifold of constant curvature $k = -1$, 0 or 1, and $f > 0$ a smooth function on an open interval I of R. A Robertson-Walker spacetime $M(k, f)$ is a warped product $I \times_f S$.

Explicitly $M(k, f)$ is the manifold $I \times S$ with the line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + f^2(t)\,\mathrm{d}\sigma^2$, where $\mathrm{d}\sigma^2$ is the line element on S lifted to $I \times S$.

Remark. The choice of this spacetime as a possible candidate for a cosmological model of the universe rests on the requirement of a spatial homogeneous and isotropic spacetime. Astronomical observations indeeed indicate that the spatial universe is approximately spherically symmetric about the earth, i.e. there is no preferred directions in the universe. The universe is said to be spatially isotropic.

Furthermore it is assumed that there is no preferred position in the universe, i.e. physical measurements do not depend on where they are performed. The universe is said to be spatially homogeneous.

Mathematically homogeneity means that the isometry group acts transitively on the space S and isotropy that the isotropy group in each point $p \in S$ acts transitively on the unit sphere of T_pS, i.e. any frame in p can be mapped to any other by the differential of an isometry. The isotropy group is the subgroup of the isometry group that leaves p fixed. It is known that every isotropic Riemannian manifold is homogeneous and complete and of constant curvature. Furthermore the odd dimensional ones are either R^n, H^n, S^n or RP^n, with n odd.

Hence the standard choices for S in a Robertson-Walker spacetime are R^3, S^3 or H^3. In coordinates t, r, θ, φ, the metric on $M(k, f)$ then takes the form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + f^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right]. \tag{23}$$

Remark. The isometric imbedding of the Robertson-Walker spacetime $M(k, f)$ into E^5, with the element

$$\mathrm{d}s^2_{E^5} = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^3 + \mathrm{d}x_4^2 - \mathrm{d}x_5^2,$$

are found in the three cases $k = 0, 1, -1$.

1) $k = 0$.

The line element on $M(k, f)$ has the form

$$ds^2 = -\,dt^2 + f^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)].$$

Now in the coordinate $x_1, \ldots, x_5$ on E^5 assume the imbedding is given by

$$x_1 = f\,r \sin\theta \cos\varphi$$

$$x_2 = f\,r \sin\theta \sin\varphi$$

$$x_3 = f\,r \cos\theta$$

$$x_4 = g_1(t, r)$$

$$x_5 = g_2(t, r).$$

Where g_1, g_2 are functions on $M(k, f)$ depending only on t, r. Then let $g_i' = \dfrac{\partial g_i}{\partial t}, \dot{g}_i = \dfrac{\partial g_i}{\partial r}$ we find:

$$ds^2 = \sum_{i=1}^{4} dx_i^2 - dx_5^2$$

$$= (r^2(f')^2 + (g_1')^2 - (g_2')^2)\,dt^2 + (rf'f + \dot{g}_1 g_1' - \dot{g}_2 g_2')\,dt\,dr$$

$$+ (f^2(\dot{g}_1)^2 - (\dot{g}_2)^2)\,dr^2 + r^2 f^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Hence the functions g_1, g_2 has to satisfy

(24) $$r^2(f')^2 + (g_1')^2 - (g_2')^2 = -1$$

(25) $$rf' + \dot{g}_1 g_1' - \dot{g}_2 g_2' = 0$$

(26) $$f^2 + (\dot{g}_1)^2 - (\dot{g}_2)^2 = f^2.$$

Setting $\alpha = g_1 - g_2$ and $\beta = g_1 + g_2$ equation (26) implies $\dot{\alpha}\dot{\beta} = 0$. Taking $\dot{\alpha} = 0$ then equations (24) and (25) take the form:

$$\alpha'\beta' = -(1 + r^2(f')^2)$$

$$\alpha'\dot{\beta} = -2rf'f.$$

If $\alpha = f$ then $\beta = -r^2 f + c(t)$ from the second equation and by inserting into the first we find

$$c(t) = -\int \frac{1}{f'}\,dt.$$

Hence we get the result:

(27) $$g_1 = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}\left((1-r^2)f - \int \frac{1}{f'}\,dt\right).$$

(28) $$g_2 = -\frac{1}{2}(\alpha - \beta) = -\frac{1}{2}\left((1+r^2)f + \int \frac{1}{f'}\,dt\right).$$

2) $k = 1$ (resp. $k = -1$).

If we set $r = \sin\chi$ (resp. $r = \sinh\chi$) the line element takes form:

$$ds^2 = -dt^2 + f^2(t)\,[d\chi^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)].$$

The imbedding is then given by:

For $k = 1$	For $k = -1$
$x_1 = f\sin\chi\sin\theta\cos\varphi$	$x_1 = f\sinh\chi\sin\theta\cos\varphi$
$x_2 = f\sin\chi\sin\theta\sin\varphi$	$x_2 = f\sinh\chi\sin\theta\sin\varphi$
$x_3 = f\sin\chi\cos\theta$	$x_3 = f\sinh\chi\cos\theta$
$x_4 = f\cos\chi$	$x_4 = \int((f')^2 - 1)^{1/2}\,dt$
$x_5 = \int((f')^2 + 1)^{1/2}\,dt$	$x_5 = f\cosh\chi$

PROPOSITION 1. *A Robertson-Walker Spacetime $M(k, f)$ can be immersed isometrically and minimally into E^5 if*

(29) $$h = f''f + 3((f')^2 + k) = 0$$

with $f'' < 0$ everywhere.

Proof. The sectional curvatures are from the metric (23) found to be:

$$K_{12} = K_{13} = K_{14} = f''/f \quad \text{and} \quad K_{23} = K_{24} = K_{34} = \frac{((f')^2 + k)}{f^2}\ .$$

so the conditions $K_{12} = K_{13}$, $K_{24} = K_{34}$ and $K_{12}K_{34} = K_{14}K_{23} = K_{13}K_{24}$ in Theorem 3 is satisfied.

The final condition is:

$$K_{23} = -\frac{1}{2}(K_{12} + K_{34}) \Rightarrow 3K_{34} + K_{12} = f''/f + 3((f')^2 + k)/f^2 = 0 \Rightarrow$$

$$h = f''f + 3((f')^2 + k) = 0.$$

Hence we find $-\lambda_1^2 = 3f''/f$ and by $h = 0, -\lambda_2^2 = -\lambda_3^2 = -\lambda_4^2 = -\dfrac{(f')^2 + k}{f^2}$ so the Ricci conditions are satisfied by (29) and $f'' < 0$ everywhere. Furthermore since $M(k, f)$ topologically is R^4, $R \times S^3$ or $R \times H^3$ they are all simply connected (and connected) hence by Theorem 3 there exists a global minimal immersion, in fact it is given by the remark on p. 13. ■

In analogy with the standard choice of stress-energy tensor in the Robertson-Walker Spacetime we define

DEFINITION. The energy-density function ρ on $M(k, f)$ is given as:

$$\rho = 3\,\frac{(f')^2 + k}{f^2}$$

Remark. The standard choice of stress-energy tensor T in Robertson-Walker spacetime is that of an ideal fluid:

$$T = (p + \rho)U^* \otimes U^* + pg,$$

where U is a timelike unit vectorfield on $M(k, f)$ and U^* is its dual and p is the pressure function on $M(k, f)$. By the Einstein equations the energy density and the pressure are then given as

$$\rho = 3K_{34};\ -p = 2K_{12} + K_{34}.$$

If $M(k, f)$ is a minimal hypersurface then by proposition 1

$$-p = 2(K_{12} + 3K_{34}) - 5K_{34} = -5/3\,K_{34} \quad \text{so}$$

$$5\,\rho = 3p.$$

Friedmann considered the case $p = 0$ in the stress-energy tensor.

The earliest era of the universe and the final one, if it exists, are dominated by radiation. There the Friedmann model gives way to radiation models, for which mass is zero and $\rho = 3p$.

For what concerns solutions to Proposition 1, $h = 0$ we have the following result.

PROPOSITION 2. *On a Robertson-Walker spacetime $M(k, f)$ with f nonconstant, the following are equivalent.*

(1) $M(k, f)$ is a minimal hypersurface.

(2) $\rho f^8 = m$ a positive constant.

(3) $(f')^2 + k = A/f^6$, where $A = m/3 > 0$.

Proof. The equivalence of (2) and (3) is immediate from the definition of ρ.

$$m = 3((f')^2 + k)f^6 \iff (f')^2 + k = (m/3)/f^6 = A/f^6.$$

If (1) holds then $h = f''f + 3((f') + k) = 0$ so $f'' = -\dfrac{3((f')^2 + k)}{f} = -\dfrac{m}{f^7}$.

Hence m is positive since by (1) $f'' < 0$. Now to show m is constant consider:

$$(mf^{-6})' = m'f^{-6} - 6mf^{-7}f'$$
$$= 6f'f'' = -6mf^{-7}f' \Rightarrow m' = 0.$$

Conversely if (2) holds then

$$0 = m' = (\rho f^8) = (3((f')^2 + k)f^6)' =$$
$$= 6f'f''f^6 + 6 \cdot 3((f')^2 + k)f^5 f' \Rightarrow hf' = 0.$$

The nonconstancy of f is needed to prove $h = 0$.

Assume h is not identical zero. Then there is a maximal interval $J \subset I$ on which h is never zero; hence $f' = 0$ on J so f is constant on J. Thus $J \neq I$. Since $h = f''f + 3((f')^2 + k))$ it is a nonzero constant on J. Thus h is nonzero on an interval strictly larger than J since h is differentiable, hence we arrived at a contradiction.

The condition $f'' < 0$ is implied by $m > 0$. ■

Remark. We find the scale function f in the three cases $k = 0, 1, -1$, setting the initial singularity at $t = 0$.

(1) $k = 0$. The equation is $f^6(f')^2 = A$ which is easily solved to $f = C\,t^{1/4}$, with $\left(\dfrac{C}{4}\right)^8 = A$.

Thus the initial expansion continues forever with $f \to \infty$ and $f' \to 0$.

(2) $k = 1$. The equation is then $(f')^2 + 1 = A/f^6$ setting

$$f = A^{1/6} \sin^{1/3} \eta(t)$$

we then find

$$t = \frac{A^{1/6}}{3}\int \sin^{1/3}\eta \, d\eta \qquad 0 < \eta < 2\pi.$$

The expansion has a maximum $f = A^{1/6}$ and a final collapse.

(3) $k = -1$. Setting

$f = A^{1/6}\sinh^{1/3}\eta(t)$ we find

$$t = \frac{A^{1/6}}{3}\int \sinh^{1/3}\eta \, d\eta.$$

Thus the universe expands forever with $f \to \infty$ and $f' \to 1$.
The graph of f is shown in the three cases in Figure 1.

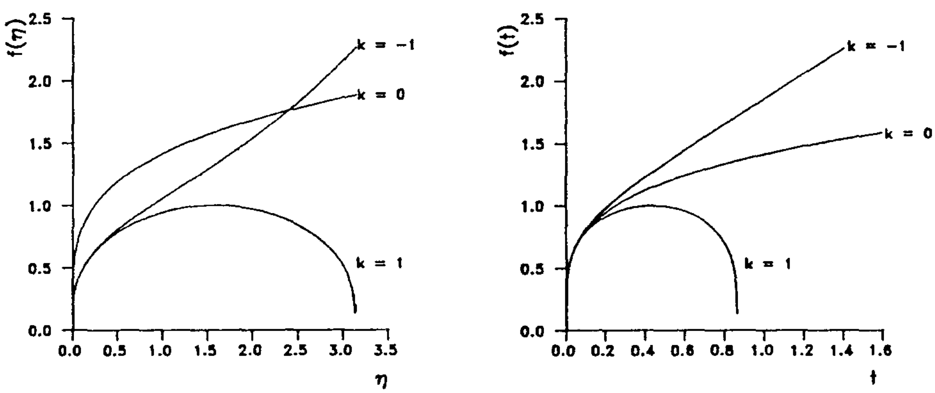

Figure 1

Remark. If we consider the 4-dimensional Riemannian analog to the Robertson-Walker spacetime $M(k, f)$, i.e. the manifold $R \times S$ with metric

$$ds^2 = dt^2 + f^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\{d\theta^2 + \sin^2\theta \, d\varphi^2\}\right].$$

Theorem 3 is still valid as noticed previously and equation (29) in Proposition 1 then takes the form

$$h = f''f + 3((f')^2 - k) = 0,$$

with $f'' > 0$. Thus equation (3) in Proposition 2 now becomes

$$(f')^2 - k = A/f^6,$$

but here A is a negative constant since $f'' > 0$, hence the cases $k = 0$ and $k = -1$ has no solutions. In the case $k = 1$ the solution is given by

$$f = a^{1/6} \cosh^{1/3} \eta(t) \qquad t = \frac{a^{1/6}}{3} \int \cosh^{1/3} \eta \, d\eta,$$

with $-A$ replaced by a; a positive constant. The solution is shown in Figure 2 and it generalizes the classical Catenoid in E^3 as the only minimal hypersurface of revolution in E^5 (for a general proof of this see [12]).

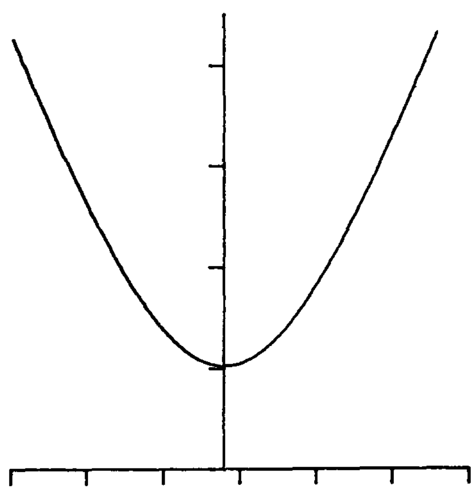

Figure 2

Remark. Astronomical data. According to Hubble (1929), all distant galaxies are moving away from us at a rate proportional to their distance. For galaxies γ_p and γ_q the distance between $\gamma_p(t)$ and $\gamma_q(t)$ in $S(t)$ is $f(t)\, d(p, q)$, where d is Riemannian distance in the space. Hubble's discovery is by current estimates:

$$H_0 = \frac{f'(t_0)}{f(t_0)} = \frac{1}{18 \pm 2 \cdot 10^9 yr} \quad (> 0).$$

With this assumption we can calculate the age of the universe — at least in the case $k = 0$ this is simple. $k = 0 \Rightarrow f = ct^{1/4}$ hence the Hubble function $H = f'/f$ is $1/4t$, hence the age of the universe is

$$t_0 = \frac{1}{4H_0} = 4{,}5\ 10^9\ yr.$$

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REFERENCES

[1] P. BAIRD, J. EELLS, *A Conservation Law for Harmonic Maps,* Lecture Notes in Mathematics, 849, 1, (1980).
[2] E. BERGER, R. BRYANT, P. GRIFFITHS, *The Gauss Eauations and Rigidity of Isometric Embeddings,* Duke Mathematical Journal, Vol. 50, No. 3, 803 (1983).
[3] C.J.S. CLARKE, *On the Global Isometric Embedding of Pseudo-Riemannian Manifolds,* Proceedings of the Royal Society of London, A 314, 417-428 (1970).
[4] J. EELLS, L. LEMAIRE, *Selected Topics in Harmonic Maps,* Bulletin of the London Mathematical Society, 10, 1, (1978).
[5] A. EINSTEIN, *Prinzipielles zur allgemeinen Relativitätstheorie,* Annalen der Physik 55, 241 (1918).
[6] A. EINSTEIN, *The Meaning of Relativity,* 4th Ed. London, 127, 127 - 128, (1950).
[7] A. EINSTEIN, *Letter to F. Pirani,* February 2, (1954).
[8] S.W. HAWKING, G.F. ELLIS, *The Large Scale Structure of Spacetime,* 75, Cambridge University Press (1973).
[9] S. KOBAYASHI, K. NOMIZU, *Foundations of Differential Geometry,* Vol. 2, 47, Interscience 1962.
[10] J. NASH, *The Embedding Problem for Riemannian Manifolds,* Annals of Mathematics Princeton, 63, 20 (1956).
[11] B. NIELSEN, *The Stress-Energy-Momentum Tensor in The Extended Nonlinear Sigma-Model,* Il Nuovo Cimento, Vol. 74 B, n. 2, 159 (1983).
[12] B. NIELSEN, *Minimal Hypersurfaces of Revolution,* preprint (1987).
[13] M. OBATA, *The Gauss Map of Immersions of Riemannian Manifolds in Space of Constant Curvature,* Journal of Differential Geometry, 2, 217 (1968).
[14] B. O'NEILL, *Semi-Riemannian Geometry With Applications to Relativity,* 204, Academic Press, 1983.
[15] A.M. POLYAKOV, *Quantum Geometry of Bosonic Strings;* Physics Letters, Vol. 103B, No. 3, 207 (1981).
[16] E.A. RUH, J. VILMS, *The Tension Fields of the Gauss Map,* Transactions of the American Mathematical Society, 149, 569-573, (1970).

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